Quiz 3.3: Sample Answers

1. Find the equation of the normal line to $y = -1 - x^2$ at $P = (3, -10)$.

   The slope of the normal line is the inverse reciprocal of the slope of the tangent line. So, we must find the slope of the tangent line at $x = 3$ using the derivative:

   $$y' = -2x$$

   So the slope of the tangent line at $x = 3$ is -6. Thus the slope of the normal line is $\frac{-1}{(-6)} = \frac{1}{6}$. We then find the equation of the line in the usual way, subsituting $x = 3$, $y = 10$, $m = \frac{1}{6}$ into $y = mx + b$ to get

   $$-10 = \frac{1}{6}(3) + b$$

   so $b = \frac{-21}{2}$. Thus the equation of the normal line is

   $$y = \frac{1}{6}x - \frac{21}{2}$$

2. There are two tangent lines to the curve $f(x) = 5x^2$ that pass through the point $P = (0, -1)$. Find the x-coordinates of the points where the tangent line intersects the curve.

   First, notice that the question is not asking us to find the equation of the tangent line at $x = 0$, but rather when the tangent line intesects the point $(0, -1)$. So, we must first find the equation of the tangent line for a general number a, then find when the equation hits that point.

   Since the derivative is $f'(x) = 10x$, the slope of the tangent line at a is $10a$. At a, the y value is $f(a) = 5a^2$. So we substitute $x = a$, $y = 5a^2$, and $m = 10a$ into $y = mx + b$ to solve for b:

   $$5a^2 = 10a(a) + b$$

   So $b = -5a^2$. Thus the equation of the tangent line to $f(x)$ at a is

   $$y = 10ax - 5a^2$$

Now we need to find where this general tangent line intersects $P = (0, -1)$. So we substitute $x = 0$, $y = -1$ into the tangent line equation:

$$-1 = 10a(0) - 5a^2$$

Then solve for a:

$$a^2 = \frac{1}{5}$$

So $a = \pm\frac{1}{\sqrt{5}}$. These are the x-coordinates of the points where the tangent line intersects the curve.

3. If a particle has equation of motion $f(t) = 2t^3 - 15t^2 + 36t + 6$, when is the particle at rest?

   A particle is at rest when its velocity is 0, so we need to solve the equation $f'(t) = 0$.

$$\begin{aligned} f'(t) &= 0 \\ 6t^2 - 30t + 36 &= 0 \\ t^2 - 5t + 6 &= 0 \\ (t-2)(t-3) &= 0 \end{aligned}$$

   Thus the particle is at rest when $t = 2$ or $t = 3$.

4. A tank holds M gallons of water which drains from the bottom of the tank in T minutes. Toricelli's Law gives the volume V of water remaining after t minutes as $V = M\left(1 - \frac{t}{T}\right)^2$, for t between 0 and T. Find the rate at which water is draining from the tank if $M = 1000$, $T = 40$, and $t = 35$.

   First, note that the values M and T are constants. So when we take the derivative of V, we are only taking it with respect to t; the M and T are just constant numbers. To take the derivative, we first expand the squared term:

$$V = M\left(1 - \frac{2t}{T} + \frac{t^2}{T^2}\right)$$

Then we take the derivative:

$$V' = M\left(\frac{-2}{T} + \frac{2t}{T^2}\right)$$

Then we can substitute our values $M = 1000$, $T = 40$, $t = 35$ to get the answer:

$$V' = 1000\left(\frac{-2}{40} + \frac{2(15)}{40^2}\right) = 168.75$$

Thus the water drains from the tank at $t = 35$ at a rate of 168.75 gallons/minute.

5. The gas law for an ideal gas at absolute temperate T (in kelvins), pressure P (in atmospheres), and volume V (in litres) is $PV = nRT$, where n is the number of moles of the gas and $R = 0.0821$ is the gas constant. Suppose that, at a certain instant, $P = 8$ atm, and is increasing at a rate which $= 0.09$ atm/min, V=10 L and is decreasing at a rate $=$ 0.14 L/min. Find the rate of change of T with respect to time at that instant if $n = 9$ mol.

   First, since we are looking for the rate of change of T, we first solve for T in the equation $PV = nRT$ to get

   $$T = n^{-1}R^{-1}PV$$

   Now, we need to take the derivative with respect to time. Note that the numbers n and R are constants, so when we take the derivative, they don't change. However, the P and V are changing with respect to time, so we need to take their derivatives. Since they are multiplied together in the equation, we use product rule to get:

   $$T' = n^{-1}R^{-1}[PV' + P'V]$$

   Then, we are given that $n = 9$, $R = 0.0821$, $P = 8$, $P' = 0.09$, $V = 10$, and $V' = -0.14$ (negative since it is decreasing). We substitute those numbers into the formula to get:

   $$T' = (9)^{-1}(0.0821)^{-1}[(8)(-0.14) + (0.09)(10)] = -0.3986$$

   Thus the temperate is changing at a rate of -0.3986 kelvins/minute.